EPFL– Fall 2025

SOLUTIONS: Series 2

Differential Geometry I: Curves and surfaces

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2.1 Prove the following formulas for the cross product: For any $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$, show that:

(i) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{b}, \mathbf{c} \rangle \mathbf{a}$ (Grassmann's 1^{st} identity),

(ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{a}, \mathbf{b} \rangle \mathbf{c}$ (Grassmann 2^{nd} identity).

(iii) $\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle \langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{a}, \mathbf{d} \rangle \langle \mathbf{b}, \mathbf{c} \rangle$ (Lagrange's identity).

(iv) $\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle = \langle (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}, \mathbf{d} \rangle$.

Hint. You might want to simplify your calculations by choosing an orthonormal basis well suited to the problem.

Solution.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ be non-zero (if one of the vectors appearing in each of the above expressions is zero, the whole expression becomes the identity $0 = 0$). It is always possible to find a positively oriented orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that

$$\begin{aligned} \mathbf{a} &= a_1 \mathbf{e}_1 \\ \mathbf{b} &= b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2, \end{aligned}$$

for example by taking $\mathbf{e}_1 = \frac{\mathbf{a}}{\|\mathbf{a}\|}, \mathbf{e}_2 = \frac{\mathbf{b} - \langle \mathbf{b}, \mathbf{e}_1 \rangle \mathbf{e}_1}{\|\mathbf{b} - \langle \mathbf{b}, \mathbf{e}_2 \rangle \mathbf{e}_2\|}$ if $\mathbf{b} - \langle \mathbf{b}, \mathbf{e}_1 \rangle \mathbf{e}_1$ is non-zero, otherwise we can take for $\mathbf{e}_2$ any unit vector perpendicular to $\mathbf{e}_1$ and finally $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Then, we express

$$\begin{aligned} \mathbf{c} &= c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3 \\ \mathbf{d} &= d_1 \mathbf{e}_1 + d_2 \mathbf{e}_2 + d_3 \mathbf{e}_3. \end{aligned}$$

In this basis, we calculate:

(i)

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= (a_1 b_2 \mathbf{e}_3) \times \mathbf{c} \\ &= -a_1 b_2 c_2 \mathbf{e}_1 + a_1 b_2 c_1 \mathbf{e}_2, \\ \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} - \langle \mathbf{b}, \mathbf{c} \rangle \mathbf{a} &= a_1 c_1 b_1 \mathbf{e}_1 + a_1 c_1 b_2 \mathbf{e}_2 - (b_1 c_1 + b_2 c_2) a_1 \mathbf{e}_1 \\ &= a_1 b_2 c_1 \mathbf{e}_2 - a_1 b_2 c_2 \mathbf{e}_1 \end{aligned}$$

(note that it is natural to expect that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ can be expressed as a linear combination of $\mathbf{a}$ and $\mathbf{b}$, since it has to be perpendicular to $\mathbf{a} \times \mathbf{b}$ which, in turn, is perpendicular to the span of $\mathbf{a}$ and $\mathbf{b}$).

(ii) Using the fact that the cross product is antisymmetric, we have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}.$$

We can then use the formula from part (i) with $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \to \{\mathbf{b}, \mathbf{c}, \mathbf{a}\}$.

(iii)

$$\begin{aligned}\langle \mathbf{a}\times\mathbf{b}, \mathbf{c}\times\mathbf{d}\rangle &= \langle a_1b_2\mathbf{e}_3, (c_2d_3-c_3d_2)\mathbf{e}_1+(c_3d_1-c_1d_3)\mathbf{e}_2+(c_1d_2-c_2d_1)\mathbf{e}_3\rangle\\ &= a_1b_2(c_1d_2-c_2d_1)\\ \langle\mathbf{a},\mathbf{c}\rangle\langle\mathbf{b},\mathbf{d}\rangle-\langle\mathbf{a},\mathbf{d}\rangle\langle\mathbf{b},\mathbf{c}\rangle &= a_1c_1(b_1d_1+b_2d_2)-a_1d_1(b_1c_1+b_2c_2)\\ &= a_1b_2c_1d_2-a_1b_2c_2d_1\end{aligned}$$

(iv)

$$\begin{aligned}\langle(\mathbf{a}\times\mathbf{b})\times\mathbf{c},\mathbf{d}\rangle &\overset{(i)}{=} \langle\langle\mathbf{a},\mathbf{c}\rangle\mathbf{b}-\langle\mathbf{b},\mathbf{c}\rangle\mathbf{a},\mathbf{d}\rangle\\ &= \langle\mathbf{a},\mathbf{c}\rangle\langle\mathbf{b},\mathbf{d}\rangle-\langle\mathbf{b},\mathbf{c}\rangle\langle\mathbf{a},\mathbf{d}\rangle\\ &\overset{(iii)}{=} \langle\mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d}\rangle\end{aligned}$$

2.2 Is the cross product on $\mathbb{R}^3$ associative? If yes, prove it, else find a counterexample.

Solution. No, the cross product is **not associative**. This can be seen via the following example: If $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a positively oriented orthonormal basis of $\mathbb{E}^3$, then

$$(\mathbf{e}_1\times\mathbf{e}_1)\times\mathbf{e}_2=\mathbf{0} \qquad \text{but} \qquad \mathbf{e}_1\times(\mathbf{e}_1\times\mathbf{e}_2)=-\mathbf{e}_2.$$

Remark. Because of the above, one should always be careful not to write an ambiguous exression such as $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$.

The Jacobi identity (see Exercise 2.3 below) in some sense measures the "non-associativity" of the cross product, since it can be rewritten as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+\mathbf{b}\times(\mathbf{a}\times\mathbf{c})$$

(using the antisymmetry of the cross product)

2.3 Show that for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{E}^3$, we have

- **(i)** $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$ (Jacobi's identity)
- **(ii)** $[\mathbf{a},\mathbf{b},\mathbf{c}]=[\mathbf{b},\mathbf{c},\mathbf{a}]=[\mathbf{c},\mathbf{a},\mathbf{b}]$,
  where $[\cdot,\cdot,\cdot]$ denotes the *mixed product.*

Remark. The Jacobi identity for the cross product, together with the fact that $\mathbf{a}\times\mathbf{b}$ is bilinear and antisymmetric in $\mathbf{a},\mathbf{b}$, implies that $(\mathbb{R}^3,\times)$ has the structure of a *Lie algebra.*

Solution.

(i) Recall the Grassmann's first identity from Ex. 2.1:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \langle\mathbf{a},\mathbf{c}\rangle\mathbf{b} - \langle\mathbf{b},\mathbf{c}\rangle\mathbf{a}.$$

We therefore have (using the symmetry of the scalar product):

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} &= \langle\mathbf{a},\mathbf{c}\rangle\mathbf{b} - \langle\mathbf{b},\mathbf{c}\rangle\mathbf{a} + \langle\mathbf{b},\mathbf{a}\rangle\mathbf{c} - \langle\mathbf{c},\mathbf{a}\rangle\mathbf{b} + \langle\mathbf{c},\mathbf{b}\rangle\mathbf{a} - \langle\mathbf{a},\mathbf{b}\rangle\mathbf{c}\\ &= \langle\mathbf{a},\mathbf{c}\rangle\mathbf{b} - \langle\mathbf{b},\mathbf{c}\rangle\mathbf{a} + \langle\mathbf{a},\mathbf{b}\rangle\mathbf{c} - \langle\mathbf{a},\mathbf{c}\rangle\mathbf{b} + \langle\mathbf{b},\mathbf{c}\rangle\mathbf{a} - \langle\mathbf{a},\mathbf{b}\rangle\mathbf{c}\\ &= 0.\end{aligned}$$

Alternative proof: Another way to derive the Jacobi identity based on a trilinearity argument is as follows. We want to prove that the map $f : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{x}\times\mathbf{y})\times\mathbf{z} + (\mathbf{y}\times\mathbf{z})\times\mathbf{x} + (\mathbf{z}\times\mathbf{x})\times\mathbf{y}$$

is identically zero. It is clear that f is trilinear, so it is sufficient to verify that f vanishes on a positively oriented orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Moreover, f is invariant (up to sign) when the variables are permuted, which reduces the number of calculations to be performed.

We first note that

$$\begin{aligned}f(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) &= (\mathbf{e}_1\times\mathbf{e}_2)\times\mathbf{e}_3 + (\mathbf{e}_2\times\mathbf{e}_3)\times\mathbf{e}_1 + (\mathbf{e}_3\times\mathbf{e}_1)\times\mathbf{e}_2\\ &= \mathbf{e}_3\times\mathbf{e}_3 + \mathbf{e}_1\times\mathbf{e}_1 + \mathbf{e}_2\times\mathbf{e}_2\\ &= 0.\end{aligned}$$

By permutation, we therefore have more generally $f(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k) = 0$ if the three indices i,j,k are pairwise distinct. When the indices are not distinct, this expression also vanishes because

$$\begin{aligned}f(\mathbf{e}_i,\mathbf{e}_i,\mathbf{e}_j) &= (\mathbf{e}_i\times\mathbf{e}_i)\times\mathbf{e}_j + (\mathbf{e}_i\times\mathbf{e}_j)\times\mathbf{e}_i + (\mathbf{e}_j\times\mathbf{e}_i)\times\mathbf{e}_i\\ &= \mathbf{0}\times\mathbf{e}_j + (\mathbf{e}_i\times\mathbf{e}_j)\times\mathbf{e}_i - (\mathbf{e}_i\times\mathbf{e}_j)\times\mathbf{e}_i\\ &= 0.\end{aligned}$$

(ii) Recall that, if $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a positively oriented orthonormal basis and $\mathbf{a} = \sum_{i=1}^3 a_i\mathbf{e}_i$, $\mathbf{b} = \sum_{i=1}^3 b_i\mathbf{e}_i$ and $\mathbf{c} = \sum_{i=1}^3 c_i\mathbf{e}_i$, then

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \det\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}$$

(it is a simple calculation to verify this expression). The required equalities now follow from the fact that the determinant of a matrix changes sign once if we flip the positions of two columns (and hence keeps the same sign under a cyclic permutation of the columns).

2.4 For any *unit* vector $\mathbf{e} \in \mathbb{E}^3$ and any $\theta \in [-\pi, \pi]$, we will define by $R^{(\mathbf{e})}_\theta : \mathbb{E}^3 \to \mathbb{E}^3$ the linear map which is the isometric rotation around the axis in the direction of $\mathbf{e}$ by the angle θ, with the orientation convention that this is a "counter clockwise" rotation (i.e. if one points their right hand thumb in the direction of $\mathbf{e}$, the rotation for $\theta > 0$ is in the direction of the rest of the fingers). Show that, for any $\mathbf{v} \in \mathbb{E}^3$, we have

$$\frac{d}{d\theta} R^{(\mathbf{e})}_\theta(\mathbf{v})\Big|_{\theta=0} = \mathbf{e} \times \mathbf{v}.$$

Remark. The above relation can be phrased as the statement that the operator $\mathbf{e}\times : \mathbb{E}^3 \to \mathbb{E}^3$ is the infinitesimal generator of the group of rotations around e. In the language of Lie groups, this will mean that the $(\mathbb{R}^3, \times)$ is the Lie algebra associated to the Lie group $SO(3)$.

Solution. Let us complete $\mathbf{e}$ into a positively oriented orthonormal basis $\{\mathbf{e}, \mathbf{w}_1, \mathbf{w}_2$. In this basis, the isometry $R^{(\mathbf{e})}_\theta$ takes the matrix form

$$R^{(\mathbf{e})}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

(note that, indeed, for $\theta > 0$, the above induces a counter-clockwise rotation in the $\{\mathbf{w}_1, \mathbf{w}_2\}$ plane). As a result, we have

$$\begin{aligned}
\frac{d}{d\theta} R^{(\mathbf{e})}_\theta(\mathbf{v})\Big|_{\theta=0} &= \frac{d}{d\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Bigg|_{\theta=0} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta) & -\cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Bigg|_{\theta=0} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -v_3 \\ v_2 \end{pmatrix}.
\end{aligned}$$

On the other hand, the right hand side above is the expression for

$$\mathbf{e} \times \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -v_3 \\ v_2 \end{pmatrix}.$$

2.5 In this exercise, we will prove a few statements about similarity transformations of a Euclidean vector space (recall the definition from first week's lecture).

(a) Prove that the similarity transformations of a Euclidean vector space $\mathbb{E}^n$ form a group.

(b) Prove that the isometries form a normal subgroup of the group of similarity transformations.

(c) Prove that the following properties are equivalent for an invertible linear map $f : \mathbb{E}^n \to \mathbb{E}^n$:

1. f is a similarity transformation.
2. f preserves angles, i.e. if $\mathbf{a}, \mathbf{b} \in \mathbb{E}^n$ are non-zero, then the angle between $f(\mathbf{a})$ and $f(\mathbf{b})$ is equal to the angle between $\mathbf{a}$ and $\mathbf{b}$.
3. f preserves orthogonality, i.e. if $\mathbf{a} \perp \mathbf{b}$ then $f(\mathbf{a}) \perp f(\mathbf{b})$.

(d) We can identify $\mathbb{C}$ with the oriented Euclidean plane $\mathbb{R}^2$. Show that $f : \mathbb{C} \to \mathbb{C}$ is a *linear* similarity transformation that preserves the orientation if and only if f is multiplication by a non-zero complex number (i.e. $f(z) = az$ with $a \in \mathbb{C}^*$). Similarly, if it changes the orientation, show that f is of the form $f(z) = a \cdot \bar{z}$. In both cases, show that f is a linear similarity transformation of scale $|a|$.

Solution. (a) Recall that a similarity transformation of a Euclidean vector space is a map $f : \mathbb{E}^n \to \mathbb{E}^n$ that satisfies $d(f(x), f(y)) = \lambda d(x, y)$, with $\lambda > 0$ independent of x and y.

Remarks: We saw in the course that a similarity transformation is an affine transformation: more precisely, a similarity transformation of $\mathbb{R}^n$ can be written as $f(x) = \lambda Ax + b$ where A is an orthogonal matrix. Two figures are *similar* if a similarity transformation maps the first figure into the second (this is why these transformations are called like that). Two figures are thus similar when all distances are multiplied by the same constant.

If $f_1, f_2 : \mathbb{E}^n \to \mathbb{E}^n$ are two similarity transformations with ratios λ_1 and λ_2, then $f_2 \circ f_1$ satisfies for all $x, y \in \mathbb{E}^n$:

$$d(f_2 \circ f_1(x), f_2 \circ f_1(y)) = d(f_2(f_1(x)), f_2(f_1(y))) = \lambda_2 d(f_1(x), f_1(y)) = \lambda_2\lambda_1 d(x, y),$$

it is therefore a similarity transformation with ratio $\lambda_1\lambda_2$. Similarly, one verifies that if f is a similarity transformation with ratio λ, then f^{-1} is a similarity transformation with ratio $1/\lambda$. We have thus shown that the similarity transformation form a subgroup of the group of bijections of the space $\mathbb{E}^n$ onto itself.

(One can also demonstrate that the similarity transformations form a group using the affine characterization $f(x) = \lambda Ax + b$ with $A \in O(n)$).

(b) The isometries are the similarity transformations with ratio 1. By the above, the composition of two isometries is an isometry and the inverse of an isometry is an isometry, so the isometries form a subgroup. This subgroup is normal because if f is a similarity transformation with ratio λ and g is an isometry, then $f \circ g \circ f^{-1}$ is a similarity transformation with ratio $\lambda \cdot \frac{1}{\lambda} = 1$, so it is an isometry.

(c) For question (e), recall that, as we have shown in class, a similarity transformation of scale $\lambda > 0$ can be written as $f(x) = \lambda g(x) + b$, where g is an isometry fixing the origin. Therefore, if f is a *linear* map, hence fixes the origin, we must have $b = 0$. In that case

$$\langle f(v), f(w)\rangle = \langle \lambda g(v), \lambda g(w)\rangle = \lambda^2 \langle v, w\rangle$$

(hence also $\|f(v)\| = \lambda\|v\|$). Using the above formula, the proof of (1) ⇒ (2) follows easily from the fact that the angle between two non-zero vectors x, y is defined by

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|\|y\|}.$$

The proof of (2) ⇒ (3) is straightforward. The proof of (3) ⇒ (1) is done as follows: let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathbb{E}^n$, and let $v_i = f(e_i)$. Then $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{E}^n$ (since we assume f is a linear invertible map) and $v_i \perp v_j$ if $i \neq j$ by hypothesis (3). We now also use that $(e_i + e_j) \perp (e_i - e_j)$, and consequently $(v_i + v_j) \perp (v_i - v_j)$. We therefore have

$$0 = \langle v_i + v_j, v_i - v_j\rangle = \langle v_i, v_i\rangle + \langle v_j, v_i\rangle - \langle v_i, v_j\rangle - \langle v_j, v_j\rangle = \|v_i\|^2 - \|v_j\|^2.$$

We have thus shown that $\|v_i\| = \|v_j\|$ for all $i, j = 1, \dots, n$. We can now prove that f is a similarity transformation with scale $\lambda = \|v_i\|$ as follows: Let x be any element of $\mathbb{E}^n$. We can write

$$x = \sum_{i=1}^n x_i e_i, \qquad \text{and therefore } \ f(x) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n x_i v_i.$$

We now determine the norm of $f(x)$ by calculating

$$\|f(x)\|^2 = \langle f(x), f(x)\rangle = \langle \sum_{i=1}^n x_i v_i, \sum_{j=1}^n x_j v_j\rangle$$

$$= \sum_{i,j=1}^n x_i x_j \langle v_i, v_j\rangle = \sum_{i=1}^n x_i^2 \|v_i\|^2 = \lambda^2 \sum_{i=1}^n x_i^2 = \lambda^2\|x\|^2,$$

(we used in this calculation that $\langle v_i, v_j\rangle = 0$ if $i \neq j$ and $\langle v_i, v_i\rangle = \|v_i\|^2 = \lambda$).

(d) Suppose the map f is written as $f(z) = az$ in complex notation with $a \neq 0$. Letting $z = x + iy$ (with $x, y \in \mathbb{R}$) and $a = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$, we directly calculate that

$$f(z) = r(\cos(\theta) + i\sin(\theta)) \cdot (x + iy) = r\left((\cos(\theta)x - \sin(\theta)y) + (\sin(\theta)x + \cos(\theta)y)i\right),$$

which can be represented in matrix form by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = r \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which is indeed a similarity transformation of scale $r = |a|$ and which preserves the orientation, since

$$\det\left(r \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\right) = r > 0.$$

To prove the converse implication, assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear similarity transformation of scale $\lambda > 0$ which preserves the orientation, then we can write $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ for some real numbers $\alpha, \beta, \gamma, \delta$. The vectors $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $\begin{pmatrix}\gamma\\ \delta\end{pmatrix}$ are the images of the canonical basis. They must therefore be orthogonal (in view of part (c)) and of the same norm, furthermore the determinant of this matrix must be positive since the similarity transformation f is assumed to preserve the orientation of the plane. We thus have the following relations:

$$\alpha\gamma + \beta\delta = 0, \qquad \alpha^2 + \beta^2 = \gamma^2 + \delta^2, \qquad \alpha\delta - \beta\gamma > 0.$$

These relations imply that $\gamma = -\beta$ and $\delta = \alpha$ and therefore $f(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$, or if we prefer in complex notation: $f(z) = (\alpha + i\beta)z$. The scale of the similarity transformation is then

$$\lambda = \frac{\left\| f\begin{pmatrix}1\\ 0\end{pmatrix}\right\|}{\left\|\begin{pmatrix}1\\ 0\end{pmatrix}\right\|} \overset{\substack{\text{In complex}\\ \text{notation}}}{=} \frac{|f(1)|}{|1|} = |\alpha + i\beta|.$$

The argument in the case when f doesn't preserve the orientation (in which case $f(x + iy) = a \cdot (x - iy)$ is completely analogous.

2.6 What are the conditions that a function $f : (0, 1) \to \mathbb{R}$ must satisfy so that its graph represents a biregular curve?

Solution. The graph of a function f represents a biregular curve if and only if the function f is of class C^2 and its second derivative f'' does not vanish.

Let's see why: Recall that a curve $\alpha : I \to \mathbb{R}^2$ is called biregular if it is of class C^2 and if $\dot\alpha(t)$ and $\ddot\alpha(t)$ are linearly independent for all $t \in I$. In the case of the graph $\gamma_f(x) = (x, f(x))$ of a function f, this means that f is of class C^2 and $f''(x) \neq 0$ for all $x \in I$. Indeed, $\dot\gamma_f(x) = (1, f'(x))$ and $\ddot\gamma_f(x) = (0, f''(x))$ are linearly independent if and only if $f''(x) \neq 0$.

2.7 Recall that the length of a curve arc $\alpha : [a, b] \to \mathbb{R}^n$ is given by

$$\ell(\alpha) = \int_a^b V_\alpha(s)ds$$

where $V_\alpha(s) = \|\dot\alpha(s)\|$ is the speed of α.

Calculate the length of the following curves:

- **(i)** $\alpha(u) = (\cos(u), \sin(u), u)$ with $-\pi \leqslant u \leqslant \pi$
  (the curve α is a right circular helix).
- **(ii)** $\beta(u) = (e^u, e^{-u}, \sqrt{2}u)$ with $0 \leqslant u \leqslant t$.
- **(iii)** $\gamma(u) = (u\cos(u), u\sin(u))$ with $0 \leqslant u \leqslant 4\pi$
  (the curve γ is an Archimedean spiral).

Solution. (a) The length of α is

$$\ell(\alpha) = \int_{-\pi}^{\pi} \|\dot{\alpha}(u)\| du = \int_{-\pi}^{\pi} \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2} du = \int_{-\pi}^{\pi} \sqrt{2} du = 2\sqrt{2}\pi.$$

For the next two questions, and in general for this course, it is important to be familiar with hyperbolic functions.

(b) The speed of the curve β is

$$V_\beta(u) = \|\dot{\beta}(u)\| = \sqrt{(e^u)^2 + (-e^{-u})^2 + (\sqrt{2})^2} = \sqrt{e^{2u} + e^{-2u} + 2} = e^u + e^{-u} = 2\cosh(u),$$

the length of this curve is therefore

$$\ell(\beta) = \int_0^t V_\beta(u) du = \int_0^t 2\cosh(u) du = 2\sinh(t).$$

(c) We have

$$\begin{aligned} \ell(\gamma) = \int_0^{4\pi} \|\dot{\gamma}(u)\| du &= \int_0^{4\pi} \sqrt{(\cos u - u\sin u)^2 + (\sin u + u\cos u)^2} du = \int_0^{4\pi} \sqrt{u^2+1} du \\ &= \frac{1}{2}\left(u\sqrt{u^2+1} + \log(u + \sqrt{u^2+1})\right)\Big|_0^{4\pi} \\ &= 2\pi\sqrt{16\pi^2+1} + \frac{1}{2}\log(4\pi + \sqrt{16\pi^2+1}). \end{aligned}$$

Remark In this last example, to find a primitive of $\sqrt{1+u^2}$ one can consult an integration table. One can also reason as follows: Let $u = \sinh(t)$, then $\sqrt{1+u^2} = \cosh(t)$ and $du = \cosh(t)dt$, so

$$\int \sqrt{1+u^2} du = \int \cosh^2(t) dt.$$

Integrate by parts:

$$\begin{aligned} \int \cosh^2(t) dt &= \int \cosh(t) \sinh'(t) dt \\ &= \cosh(t)\sinh(t) - \int \sinh(t)\cosh'(t) dt \\ &= \cosh(t)\sinh(t) - \int \sinh^2(t) dt \\ &= \cosh(t)\sinh(t) - \int \left(\cosh^2(t) - 1\right) dt. \end{aligned}$$

Consequently

$$\int \cosh^2(t)dt = \frac{1}{2}\left(t + \sinh(t)\cosh(t)\right) + C = \frac{1}{2}\left(t + \sinh(t)\sqrt{1+\sinh(t)^2}\right) + C,$$

where C is a constant of integration. Finally,

$$\begin{aligned}\int \sqrt{1+u^2}du &= \int \cosh^2(t)dt \\ &= \frac{1}{2}\left(t + \sinh(t)\sqrt{1+\sinh(t)^2}\right) + C \\ &= \frac{1}{2}\left(\log\left(u+\sqrt{u^2+1}\right) + u\sqrt{1+u^2}\right) + C,\end{aligned}$$

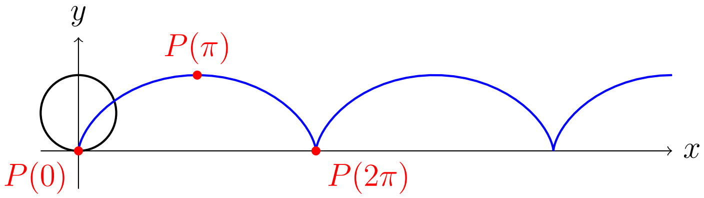
because $t = \sinh^{-1}(u) = \operatorname{arcsinh}(u) = \log\left(u+\sqrt{u^2+1}\right)$.

2.8 The cycloid is the curve traced out by a point on the edge of a wheel that rolls, without slipping, in a straight line.

(a) Draw a cycloid. Is it a smooth curve? Write down a parametrization of the cycloid.

(b) Calculate the length of one arch of the cycloid (assuming the wheel generating the cycloid has radius r).

Solution. (a)



Let P be a point on the edge of a wheel of radius r and center C. We will roll the wheel on the x-axis. At time $t = 0$, we place the wheel so that the point P coincides with the origin. After a time t, the radius passing through P will form an angle t with the vertical radius. The wheel will have traveled a distance rt. We wish to calculate the coordinates of the point P as a function of t. We have

$$\vec{OP} = \vec{OA} + \vec{AC} + \vec{CP} = \begin{pmatrix} rt \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ r \end{pmatrix} + \begin{pmatrix} -r\sin t \\ -r\cos t \end{pmatrix}$$

where A is the (moving) point of intersection of the wheel with the x-axis. Hence a parametrization of the cycloid:

$$\gamma(t) = (r(t-\sin t), r(1-\cos t)).$$

(c) The velocity vector is $\dot\gamma(t) = (r(1-\cos t), r\sin t)$ and the speed is $\|\dot\gamma(t)\| = r\sqrt{(1-\cos t)^2 + (\sin t)^2} = r\sqrt{2(1-\cos t)}$. The length of one arch of the cycloid is therefore

$$L = \int_0^{2\pi} \|\dot\gamma(t)\|dt = r\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos t}dt = r\sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2(t/2)}dt$$

$$= 2r\int_0^{2\pi} |\sin(t/2)|dt = 2r\int_0^{2\pi} \sin(t/2)dt = 2r\cdot[-2\cos(t/2)]_{t=0}^{2\pi} = 8r.$$

Note that it we can replace $\sqrt{\sin^2(t/2)}$ with $\sin(t/2)$ because $\sin(t/2)\geqslant 0$ for $t\in[0,2\pi]$.